ChaBONNty conference lightning talks
(Algorithmic modular curve Chabauty-Coleman without equations)

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Mazur's Program B states:

> *Suppose $E/\mathbf{Q}$ is non-CM. What are the possible images of $\rho_E \colon \mathrm{Gal}(\overline{\mathbf{Q}}/\mathbf{Q}) \to \mathrm{GL}_2(\widehat{\mathbf{Z}})$, the Galois representation associated to the adelic Tate module of E?*

This problem can be recast as finding rational points on various modular curves.

- $X(1)$ parametrizes elliptic curves E.
- $X(N) \to X(1)$ parametrizes level N structures $\left[\begin{smallmatrix} P \\ Q \end{smallmatrix}\right] : \mathbf{Z}(N)^2 \to E[N]$ "up to isomorphism".
- For $G \leq \mathrm{GL}_2(N)$, $X_G \to X(1)$ parametrizes left G-orbits of level N structures "up to geometric isomorphism".
- Thus, $X_G(\mathbf{Q})$ gives the data of $E_{/\mathbf{Q}}$, plus a level N structure witnessing the mod N image lying in G, "up to twist".

You will be seeing a lot of this over the next few days.

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Work of Zywina breaks Mazur's Program B into three steps:

1. Solve Serre's uniformity conjecture.
2. Determine the rational points on X_G, for a specific set of 841 groups G.
3. Analyze the structure of 160 different "twist-parametrized families".

We will focus on Step 2 for curves that satisfy the Chabauty-Coleman condition.

- More about Step 1 in Jan's lecture.
- More about Step 2 beyond Chabauty-Coleman in Guido's and Sachi's lectures. (These lectures will use roughly the same strategy that I am about to outline.)
- More about Step 3 in Filip's lecture. (The work presented in his lecture greatly clarifies Zywina's strategy.)

Theorem (X.)

Suppose $G \leq \mathrm{GL}_2(N)$ contains all scalar matrices, and suppose the Chabauty-Coleman condition is satisfied for X_G. Then there is an explicit and relatively efficient algorithm to compute $X_G(\mathbf{Q}_p)_1$.

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This is Kamal Khuri-Makdisi. You may or may not have heard of him, so here is some lore.

- Kamal Khuri-Makdisi was a student of Shimura; he's now situated in Lebanon.
- I'm told he taught my PhD advisor at one point.
- He's quite pleasant to talk to! (At least pleasant enough to allow me to shorten his name to "Makdisi" in what follows...)

Kamal Khuri-Makdisi is arguably the person whose work will be the most important for the study of rational points on modular curves. He is the reason that my thesis exists.

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The key idea that makes everything hold together is the use of a gadget that I dub a *Makdisi symbol*. (It's just a bunch of Eisenstein series.)

Definition (Makdisi symbol)

For a matrix $\gamma = \left[\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right] \in \mathrm{Mat}_{2\times 2}(\mathbf{Z}/N\mathbf{Z})$, let $\mathrm{Mak}_N(\gamma)$ denote the cuspidal projection of $E_1((a,b)_{/N})E_1((c,d)_{/N})$. This lives on $X(N)$. Also define $\mathrm{Mak}_G(\gamma) := \sum_{g \in G/\pm 1} \mathrm{Mak}_N(\gamma \cdot g)$.

Here are the key properties of Makdisi symbols.

- Makdisi symbols span the space of weight 2 cusp forms.
- If we restrict matrix γ to be *invertible*, then Makdisi symbols span the space of weight 2 cusp forms associated to the rank 0 part of $\mathrm{Jac}(X_G)$.
- Makdisi symbols satisfy the same 2- and 3-term Manin relations that you see if you are used to studying modular symbols.
- The action of the Hecke operator T_p on $\mathrm{Mak}_G(\gamma)$ can be expressed in $O(p\log(p))$ Makdisi symbols of the form $\mathrm{Mak}_G(X \cdot \gamma)$ for matrices X only dependent on p. (These resemble Merel's "universal matrices"...)
- Weights 1 and 2 Eisenstein series on $X(N)$ have very nice moduli interpretations in terms of the coordinates the N-torsion points of the universal elliptic curve. (Whence power series at arbitrary points!)

All but the second item is implicit in Khuri-Makdisi's paper "Moduli interpretation of Eisenstein series".

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Outline of the proof

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Definition (Makdisi symbol)

For a matrix $\gamma = \left[\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right] \in \mathrm{Mat}_{2\times 2}(\mathbf{Z}/N\mathbf{Z})$, let $\mathrm{Mak}_N(\gamma)$ denote the cuspidal projection of $E_1((a,b)_{/N})E_1((c,d)_{/N})$. This lives on $X(N)$. Also define $\mathrm{Mak}_G(\gamma) := \sum_{g\in G/\pm 1} \mathrm{Mak}_N(\gamma\cdot g)$.

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- The code runs *reasonably* fast, taking 10 minutes to compute $X_G(\mathbf{Q}_p)_1$ to around 100 p-adic digits of precision for the curve `10.60.2.f.1` at the prime $p = 11$.
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