ON SKELET #17 AND THE DETERMINATION OF BB(5)

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1. Description

1.1. **Introduction.** The *busy beaver* function $\mathrm{BB}(n)$ is defined to be the maximum number of moves a halting n-state 2-symbol Turing machine makes before it halts. Although $\mathrm{BB}(n)$ is uncomputable for general n, it has long been conjectured that $\mathrm{BB}(5) = 47{,}176{,}870$, and large amounts of effort have been spent trying to rule out any halting machines that might make more than 47.176,870 moves before terminating.

A massive undertaking by The Busy Beaver Challenge team has narrowed the determination of $BB(5)$ down to six machines; the rest have been shown to not halt or terminate before 47,176,870 steps. Of the six machines, five have been classified as "cubic finned" machines and are expected to have relatively simple non-halting proofs. By contrast, the final machine, Skelet #17, has eluded all known attempts at classification.

1.2. **Skelet #17.** It is known that Skelet #17 can be described by the following process: begin with the state $S = (0, 2, 4, 0)$. Let $P(S)$ be the next state after S. $P(S)$ is defined by the following rules:

- Overflow: If $S = (2a_\ell + 1, 2a_{\ell-1}, \ldots, 2a_0)$, transition to $P(S) = (0, 2a_\ell + 2, 2a_{\ell-1}, \ldots, 2a_0)$.
- Halt: If $S = (0, 0, 2a_{\ell-2}, \ldots, 2a_0)$, halt.
- Zero: If $S = (2a_\ell, 2a_{\ell-1}, \ldots, 2a_0)$ and $(a_1, a_2) \neq (0, 0)$, transition to $P(S) = (0, 0, 2a_\ell+1, 2a_{\ell-1}, \ldots, 2a_1, 2a_0-1)$.
- Halve: If $S = (a_\ell, \ldots, a_1, -1)$, transition to $P(S) = (a_\ell, \ldots, a_1)$.
- Increment: If $S = (a_\ell, \ldots, a_1, a_0)$ is not in the form specified by any of the above rules, find the rightmost index a_i of S with an odd value. Transition to $P(S) = (a_\ell, a_{i+2}, a_{i+1} + 1, a_i, \ldots, a_1, a_0 - 1)$.

Remark 1.1. Note that the conventions of savask's document are slightly different from ours: what corresponds to "Overflow" there is "Overflow + Zero" here, and what corresponds to "Halve" there is "Increment + Halve" here.

Definition 1.2. Given two states S and T, say that $S \mapsto T$ if $P^k(S) = T$ for some $k \in \mathbb{N}$. In this case, define $S \to T$ to be the sequence of transition rules that were applied from S to obtain T.

The goal of this paper is to show that the above process never terminates. In particular, we will establish the following theorem:

Theorem 1.3. We have $(0, 2, 4, \ldots, 2^{2k}, 0) \mapsto (0, 2, 4, \ldots, 2^{2k+2}, 0)$ for all $k \in \mathbb{N}$. Moreover, during this, we use exactly one Overflow rule. As a result, Skelet #17 never halts.

Conditioning on non-halting of the "cubic-finned" machines, we can therefore conclude:

Corollary 1.4. $\mathrm{BB}(5) = 47, 176, 870$.

2. Basics

2.1. **State variables.** For $r \in \mathbb{R}$, let $\langle r \rangle$ denote the nearest integer to r, where we round up if r is a half-integer (e.g $\langle 1.5 \rangle = 2$).

Definition 2.1. For $n \in \mathbb{N}$, the *Gray code* of n is defined to be $\mathrm{GrayCode}(n) := \cdots a_3 a_2 a_1 a_0$, where each digit $a_i \in \{0, 1\}$ equals the mod 2 reduction of $\langle n/2^i \rangle$.

For a state $S = (a_\ell, \ldots, a_0)$, introduce the following notation, which is specified in order to capture the most important properties of Skelet #17:

- Let $n := n(S)$ denote the number $\mathrm{GrayCode}^{-1}(\overline{a_\ell a_{\ell-1}} \ldots \overline{a_1})$. Here we let $\overline{a_i} := a_i \bmod 2$, and note that we are not considering a_0 in the formation of n.

Date: February 27, 2024.

- Let $\ell := \ell(S)$ denote the number one less than the size of S, i.e. $|S| - 1$.
- Let $\sigma := \sigma(S) \in \{-1, +1\}$ be $+1$ if $\sum_{i=0}^{\ell} a_i$ is odd, and -1 if $\sum_{i=0}^{\ell} a_i$ is even.
- In addition to the three state variables above, let $a_i := a_i(S)$ denote the value a_i of the i^{th} index of S, for $i \in [0, \ell]$.

Under the transition rules, we may write down how the first three state variables change for each rule that is applied. In particular, one can verify the following table (where the substitutions in a given rule's row are performed going left to right):

Rule	n	ℓ	σ
Overflow	$2^\ell - 1 \mapsto 0$	$\ell \mapsto \ell + 1$	$+1 \mapsto -1$
Empty	$0 \mapsto 2^\ell - 1$	$\ell \mapsto \ell + 2$	$-1 \to -1$
Halve	$n \mapsto \lfloor n/2 \rfloor$	$\ell \mapsto \ell - 1$	$\sigma \mapsto -\sigma$
Increment	$n \mapsto n + \sigma$	$\ell \mapsto \ell$	$\sigma \mapsto \sigma$
Halt	$0 \mapsto$ N/A	$\ell \mapsto$ N/A	$-1 \mapsto$ N/A

The above table does more than just specify how the state variables change with each rule: it also restricts what (n, ℓ, σ) can be immediately before a given rule is applied. For instance, if S is a state for which we Overflow out of, then we must necessarily have $n(S) = 2^{\ell(S)} - 1$ and $\sigma(S) = +1$.

2.2. Increments.

Define the function

$$d_j(a, b) := \left| \left\langle \frac{a}{2^j} \right\rangle - \left\langle \frac{b}{2^j} \right\rangle \right|.$$

The next proposition states how a state's coordinates change under a series of increments:

Proposition 2.2. Suppose we have states S and S' where $S \mapsto S'$, such that $S \to S'$ consists of rules of the form Increment. Denote $n := n(S)$, $n' := n(S')$ and $\sigma := \sigma(S) = \sigma(S')$. Then for each $i \in [0, \ell]$ we have

$$a_i(S') = \begin{cases} a_i(S) + d_i(n, n') & i \geq 1 \\ a_i(S) - d_i(n, n') & i = 0. \end{cases}$$

Proof. For $i = 0$, we have $d_i(n, n') = |n' - n|$, which is precisely the number of Increment rules in $S \to S'$. Since a_0 decreases after an Increment rule, the claim $d_0(n, n') = a_0(S) - a_0(S')$ follows immediately. For $i > 0$, note that by Definition 2.1, the value $d_i(n, n')$ quantifies how much a_i increases when incrementing the Gray code of n to the Gray code of n'. □

3. THE SPEEDUP

3.1. Conventions.

From this section onwards, fix $k \in \mathbb{N}$ and let $S_k := (0, 2, 2^2, \ldots, 2^{2k}, 0)$. From now on, every state E we will consider will satisfy $S_k \mapsto E$, but not $S_{k+1} \mapsto E$. For $E = (a_\ell, \ldots, a_1, a_0)$ and $i \in [0, \ell]$, denote $E[i] := (a_\ell, \ldots, a_{i+1}, a_i + 2, a_{i-1}, \ldots, a_0)$, which is E but with a_i incremented by $+2$.

3.2. Empty and embanked states.

Definition 3.1. Call a state E *empty* if $n(E) = 0$ and $\sigma(E) = -1$, but the next rule applied is not Halt (so in particular the state immediately after E is achieved by applying Zero).

If E is empty, then let $N(E) := T_E(E)$ denote the next state after E that is empty. Let T_E denote the sequence of rules $E \to N(E)$ if $N(E)$ exists, and otherwise let T_E denote the sequence of rules after E.

Definition 3.2. Let E be an empty state.

(1) Say that E is *embanked* if the non-Increment rules of T_E consist of exactly one Zero rule at the start and exactly two Halve rules elsewhere.
(2) Say that E is *weakly embanked* if the rules of T_E consist of one Zero rule at the start and at least two Halve rules (in particular, there may be arbitrarily many), such that all other rules before the second Halve rule are Increment rules.
(3) Suppose E is weakly embanked. For $i \in \{1, 2\}$, define $h_i := h_i(E)$ (resp. $s_i := s_i(E)$) to be the value of n for the state immediately after (resp. before) the i^{th} Halve rule is applied in T_E. Let $h := h(E)$ (resp. $s := s(E)$) denote the tuple $(h_1(E), h_2(E))$ (resp. $(s_1(E), s_2(E))$). So in particular we have $h_i = \lfloor s_i/2 \rfloor$.

In the course of proving Theorem 1.3, we will in fact see that the vast majority of the empty states we consider are embanked. In the first non-embanked empty state we see, an Overflow rule results, at which point an explicit analysis will yield the desired result. It is in this fact that the reason for the term *embanked* becomes clear: it signifies that T_E flows extremely well as we repeatedly apply $N(\cdot)$, and in particular well enough for us to apply major speedups to the Skelet #17 process.

Example 3.3. Let $E := (2,2,6,8,18,0)$. Then E is embanked with $h(E) = (15,17)$. Let us spell out T_E in detail: n will start at 31 after the first Empty rule, with corresponding state $(0,0,3,2,6,8,18,-1)$. Here, we have $a_0 = -1$, so we must immediately apply the first Halve rule, which takes n to $\lfloor 31/2 \rfloor = 15$ and σ to $+1$, with ensuing state $(0,0,3,2,6,8,18)$. A series of Increment rules are applied until $n = 34$, where the state is now $(1,1,4,4,11,17,-1)$. Then the second Halve rule is applied to yield $(n,\sigma) = (17,-1)$, and after a further series of Increment rules we have $n = 0$, which corresponds to the state $E' = (2,2,6,8,20,0) = E[1]$. The reader is highly encouraged to graph the values of n attained in T_E to gain an intuition behind this paper.

Proposition 3.4. Empty state E is weakly embanked if and only if the following conditions hold:

(1) $a_0(E) < 2^{2k+1} - 1$.
(2) $a_1(E) < 3 \cdot 2^{2k} - 1 - \overline{a_0(E)}$,

where as in Definition 2.1, we denote $\overline{a_0(E)} \in \{0,1\}$ as the mod 2 reduction of $a_0(E)$.

Proof. For E to be weakly embanked, the sequence $T_{\underline{\text{Zero}}(E)}$ must have two Halve rules before any non-Increment rule. To guarantee that the first Halve rule occurs before any non-Increment rule, the variable a_0 must decrement to -1 from $a_0(\underline{\text{Zero}}(E)) = a_0(E) - 1$ before the variable n decrements to 0 from $n(\underline{\text{Zero}}(E)) = 2^{2k+1} - 1$. This is the same as the first condition.

Assume condition (1) holds, and let E' be the state immediately after applying the first Halve rule in T_E. To guarantee that the second Halve rule occurs before any non-Increment rule, the variable a_0 must decrement from $a_0(E')$ to -1 before the variable n increments from $n(E')$ to $2^{2k+2} - 1$ (after which an Overflow rule would have to be applied). We readily compute

$$a_0(E') = a_1(E) + d_1(2^{2k+1} - 1, 2^{2k+1} - 1 - a_0)$$
$$n(E') = \left\lfloor \frac{2^{2k+1} - 1 - a_0(E)}{2} \right\rfloor$$

so that the desired condition becomes the inequality

$$a_1(E) + d_1(2^{2k+1} - 1, 2^{2k+1} - 1 - a_0(E)) + 1 < 2^{2k+2} - 1 - \left\lfloor \frac{2^{2k+1} - 1 - a_0(E)}{2} \right\rfloor.$$

The parity of $a_0(E)$ affects the simplification of the above inequality: namely, it is equivalent to

$$\begin{cases} a_1(E) + \frac{a_0(E)+1}{2} + 1 < 2^{2k+2} - 1 - \left(2^{2k} - \frac{a_0(E)+1}{2}\right) & a_0(E) \equiv 1 \bmod 2 \\ a_1(E) + \frac{a_0(E)}{2} + 1 < 2^{2k+2} - 1 - \left(2^{2k} - \frac{a_0(E)+2}{2}\right) & a_0(E) \equiv 0 \bmod 2, \end{cases}$$

which simplifies to

$$a_1(E) < \begin{cases} 3 \cdot 2^{2k} - 2 & a_0(E) \equiv 1 \bmod 2 \\ 3 \cdot 2^{2k} - 1 & a_0(E) \equiv 0 \bmod 2. \end{cases}$$

This is exactly condition (2), so we win. □

3.3. Speedup.

Lemma 3.5. Let $E = (a_\ell, \dots, a_1, a_0)$ be an embanked state with $h(E) =: (h_1, h_2)$, $s(E) =: (s_1, s_2)$, and choose $i \in [0, \ell]$ such that $E[i]$ is weakly embanked. Then we have

$$h(E[i]) = \begin{cases} (h_1 - 1, h_2) & i = 0 \\ (h_1, h_2 + 1) & i = 1 \\ (h_1, h_2) & i \geq 2. \end{cases}$$

$$s(E[i]) = \begin{cases} (s_1 - 2, s_2) & i = 0 \\ (s_1, s_2 + 2) & i = 1 \\ (s_1, s_2) & i \geq 2. \end{cases}$$

Proof. Note that $h(E)$ depends on only the indices a_1 and a_0 of E (this immediately implies $h(E[i]) = h(E)$ for $i \geq 2$). In particular, a_0 influences h_1 while a_1 influences h_2. Thus, if $i = 0$, then, before the first Halve rule of $T_{E[i]}$, we must apply the Increment rule two additional times compared to T_E before the necessary $a_0 = -1$ occurs; it follows that $h(E[0]) = (h_1 - 1, h_2)$. A similar analysis for the states preceding the second Halve rule of $T_{E[i]}$ yields $h(E[1]) = (h_1, h_2 + 1)$. □

Proposition 3.6. Suppose E is an embanked state such that $N(E) = E[i]$ for some $i \in [0, \ell(E)]$, and suppose $N(E)$ is weakly embanked. Let $\ell = \ell(E)$, $h(E) = (h_1, h_2)$ and $s(E) = (s_1, s_2)$. Then $N(E)$ is in fact embanked, with

$$N(N(E)) = \begin{cases} N(E)[\nu_2(h_1)] & i = 0 \\ N(E)[\nu_2(h_2 + 1) + 1] & i = 1 \\ N(E)[i - 2] & i \geq 2. \end{cases}$$

Proof. If $i \geq 2$, then by Lemma 3.5, $h(N(E)) = h(E)$. $N(E)$ is just E with a_i incremented by $+2$, and the i^{th} index will shift to the $(i-2)^{th}$ index upon another application of N. Since the indices a_1 and a_0 are the same for E and $N(E)$, we have $T_E = T_{N(E)}$, hence $N(E)$ is embanked with $N(N(E)) = N(E)[i - 2]$.

The nontrivial cases are $i \in \{0, 1\}$. If $i = 1$, then Lemma 3.5 implies that, compared to T_E, the transition $T_{E[1]}$ acquires exactly two additional Increment rules between the first and second Halve rules, as well as one additional Increment rule after the second Halve rule. Let us analyze what happens to the j^{th} index for each $j \in [0, \ell]$. Let T'_E denote the transition $\underline{\text{Zero}}(E) \to N(E)$; this is T_E but with the first transition removed. By Proposition 2.2, we have

$$a_j(E[1]) = \begin{cases} a_{j+2}(\underline{\text{Zero}}(E)) + d_{j+2}(2^\ell - 1, s_1) + d_{j+1}(h_1, s_2) + d_j(h_2, 0) & j \geq 1 \\ a_{j+2}(\underline{\text{Zero}}(E)) + d_{j+2}(2^\ell - 1, s_1) + d_{j+1}(h_1, s_2) - d_j(h_2, 0) & j = 0, \end{cases}$$

noting that (1) the indices shift every time a Halve rule is applied, and (2) all d_j's are in fact defined, since Zero adds two extra indices to the state. Moreover, define

$$a'_j(N(E[1])) := \begin{cases} a_{j+2}(\underline{\text{Zero}}(E[1])) + d_{j+2}(2^\ell - 1, s_1) + d_{j+1}(h_1, s_2 + 2) + d_j(h_2 + 1, 0) & j \geq 1 \\ a_{j+2}(\underline{\text{Zero}}(E[1])) + d_{j+2}(2^\ell - 1, s_1) + d_{j+1}(h_1, s_2 + 2) - d_j(h_2 + 1, 0) & j = 0, \end{cases}$$

which agrees with $a_j(N(E[1])$ if $N(E) = E[1]$ is embanked, but will still make sense if $N(E)$ is only weakly embanked. Note that $N(E)$ will fail to be embanked if and only if $a'_0(N(E[1])) < 0$; in this case, for $T_{E[1]}$ after the second Halve rule, the quantity a_0 decrements to -1 before n can decrement to 0, at which point a third Halve rule has to be applied. From the above expressions, we obtain

$$\begin{aligned} a'_j(N(E[1])) - a_j(E[1]) &= \begin{cases} (d_{j+1}(h_1, s_2 + 2) - d_{j+1}(h_1, s_2)) + (d_j(h_2 + 1, 0) - d_j(h_2, 0)) & j \geq 1 \\ (d_{j+1}(h_1, s_2 + 2) - d_{j+1}(h_1, s_2)) - (d_j(h_2 + 1, 0) - d_j(h_2, 0)) & j = 0 \end{cases} \\ &= \begin{cases} d_{j+1}(s_2 + 2, s_2) + d_j(h_2 + 1, h_2) & j \geq 1 \\ d_{j+1}(s_2 + 2, s_2) - d_j(h_2 + 1, h_2) & j = 0 \end{cases} \\ &= \begin{cases} 2d_j(h_2 + 1, h_2) & j \geq 1 \\ 0 & j = 0 \end{cases} \end{aligned}$$

where the first equality is justified by the fact that $a_{j+2}(\underline{\text{Zero}}(E[1])) = a_{j+2}(\underline{\text{Zero}}(E))$ by the definition of $E[i]$, and the third equality is justified by verifying that $d_{j+1}(s_2 + 2, s_2) = d_j(h_2 + 1, h_2)$ always holds. We learn $a'_j(N(E[1])) - a_j(E[1]) \geq 0$, and therefore that $N(E)$ is embanked. Moreover, we have exactly one index $j \in [0, \ell(E)]$

such that $N(E[1]) = E[1][j]$; this is the index for which $d_j(h_2+1, h_2) > 0$. Such j is characterized by $(h_2+1)/2^j$ being a half-integer, and this occurs precisely when $\nu_2(h_2+1) = j-1$. The desired claim follows for the $i=1$ case.

If $i = 0$, a completely analogous analysis occurs: we compute

$$a_j(E[0]) = \begin{cases} a_{j+2}(\underline{\text{Zero}}(E)) + d_{j+2}(2^\ell - 1, s_1) + d_{j+1}(h_1, s_2) + d_j(h_2, 0) & j \geq 1 \\ a_{j+2}(\underline{\text{Zero}}(E)) + d_{j+2}(2^\ell - 1, s_1) + d_{j+1}(h_1, s_2) - d_j(h_2, 0) & j = 0, \end{cases}$$

and define

$$a'_j(N(E[0])) := \begin{cases} a_{j+2}(\underline{\text{Zero}}(E[0])) + d_{j+2}(2^\ell - 1, s_1 - 2) + d_{j+1}(h_1 - 1, s_2) + d_j(h_2, 0) & j \geq 1 \\ a_{j+2}(\underline{\text{Zero}}(E[0])) + d_{j+2}(2^\ell - 1, s_1 - 2) + d_{j+1}(h_1 - 1, s_2) - d_j(h_2, 0) & j = 0, \end{cases}$$

so that for all $j \in [0, \ell]$, we have

$$\begin{aligned} a'_j(N(E[0])) - a_j(E[0]) &= d_{j+2}(s_1 - 2, s_1) + d_{j+1}(h_1 - 1, h_1) \\ &= 2d_{j+1}(h_1 - 1, h_1). \end{aligned}$$

As in the $i = 1$ case, the fact that $a'_j(N(E[0])) - a_j(E[0]) \geq 0$ immediately implies that $N(E)$ is embanked, and in fact $a'_j(N(E[0])) - a_j(E[0])$ positive if and only if $h_1/2^{j+1}$ is a half-integer, which occurs precisely when $\nu_2(h_1) = j$. This completes the proof. □

In light of Proposition 3.6, we are induced to *speed up* the function $N(\cdot)$ to a new function $N'(\cdot)$:

Definition 3.7. Define the following terms:

(1) For an embanked state E such that $N(E) = E[i]$, define $N'(E)$ to be the state

$$\begin{aligned} N'(E) &:= N^{\lceil (i(E)+1)/2 \rceil}(E) \\ &= \begin{cases} E[i][i-2]\cdots[3][1] & i \text{ odd} \\ E[i][i-2]\cdots[2][0] & i \text{ even}, \end{cases} \end{aligned}$$

where the second equality holds by Proposition 3.6.

(2) Say that an embanked state E is *rooted embanked* if it is the result of applying N' to S_k some amount of times, i.e. there exists $e \in \mathbb{N}$ such that $E = (N')^e(S_k)$.

(3) For $i \in \{0, 1\}$, say that a rooted embanked state E is *i-rooted embanked* if $E = N^{-1}(E)[i]$. Every rooted embanked state is either 0-rooted embanked or 1-rooted embanked.

(4) For a state transition $E \to E'$ of rooted embanked states, let $T'_{E \to E'}$ denote the sequence of N's that were applied to E to achieve E'.

We have the immediate corollary:

Corollary 3.8. For $i \in \{0, 1\}$, let E be an i-rooted embanked state with $h(N^{-1}(E)) = (h_1, h_2)$, and assume that $N(E)$ is weakly embanked. Then we have

$$N(E) = \begin{cases} E[\nu_2(h_1)] & i = 0 \\ E[\nu_2(h_2 + 1) + 1] & i = 1, \end{cases}$$

so that $N'(E)$ is $\overline{\nu_2(h_1)}$-rooted embanked if $i = 0$, and $\overline{\nu_2(h_2+1)+1}$-rooted embanked if $i = 1$.

4. Proof of nonhalting

Proposition 4.1. Let E be a 1-rooted embanked state such that there exists $m \in [0, 2^{2k} - 2)$ of odd 2-adic valuation such that $h(N^{-1}(E)) = (2^{2k} - m - 1, 2^{2k} + m)$. Then the following hold:

(1) If $m' > m$ in $[0, 2^{2k} - 2]$ denotes the next number after m satisfying $\nu_2(m') \equiv 1 \bmod 2$, then there exists a unique 1-rooted embanked state E' satisfying $E \mapsto E'$, such that $h(N^{-1}(E')) = (2^{2k} - m' - 1, 2^{2k} + m')$.

(2) In the setting of (2), let $d := m' - m$. Then $a_i(E') = a_i(E) + 2d$ for $i \in \{0, 1\}$.

Proof. Suppose E and m are as in the problem statement. In what follows, the quantities a_1 and a_0 never breach the bounds given by the conditions in Proposition 3.4; combining this with Corollary 3.8, it follows that all N'-iterates of E we consider will be rooted embanked.

We claim that for each $e \in [0, m' - m - 1]$, $E_e := (N')^e(E)$ is 1-rooted embanked with $h(N^{-1}(E_e)) = (2^{2k} - m - 1, 2^{2k} + m + e)$. This follows by induction on e: the base case $e = 0$ is given, while for the induction step, if $e < m' - m - 1$ satisfies the induction hypothesis, then since $m + e + 1 < m'$, it must necessarily have even valuation; hence by Corollary 3.8, $E_{e+1} := N'(E_e)$ is $\overline{\nu_2((m+e)+1)+1} = 1$-rooted embanked, and moreover by Lemma 3.5 we have $h(N^{-1}(E_{e+1})) = h(N^{-1}(E_e)[1]) = (2^{2k} - m - 1, 2^{2k} + m + e + 1)$. This completes the induction. In addition, we also learn that from E to $E_{m'-m-1}$, the quantity a_1 increments by $2(m' - m - 1)$.

When we apply N' to $E_{m'-m-1}$, we obtain a state E' such that $h(N^{-1}(E')) = (2^{2k} - m - 1, 2^{2k} + m')$. Since $\nu_2((m'-1)+1) = \nu_2(m')$ is odd, it follows by Corollary 3.8 that $E' = N'(E_{m'-m-1})$ is $\overline{\nu_2((m'-1)+1)+1} = 0$-rooted. We claim that for each $e \in [0, m' - m - 1]$, $E'_e := (N')^e(E')$ is 0-rooted embanked with $h(N^{-1}(E'_e)) = (2^{2k} - m - 1 - e, 2^{2k} + m')$. This, again, follows by induction on e: the base case $e = 0$ is given, while for the induction step, if $e < m' - m - 1$ satisfies the induction hypothesis, then since $m + e + 1 < m'$, it must necessarily have even valuation; hence by Corollary 3.8, $E'_{e+1} := N'(E'_e)$ is $\overline{\nu_2(2^{2k} - m - 1 - e)} = 0$-rooted embanked, and moreover by Lemma 3.5 we have $h(N^{-1}(E'_{e+1})) = h(N^{-1}(E'_e)[0]) = (2^{2k} - m - 2 - e, 2^{2k} + m')$. This completes the induction. In addition, we also learn that from $E_{m'-m-1}$ to $E'_{m'-m-1}$, the quantity a_0 decrements by $2(m' - m)$.

Finally, applying N' to $E'_{m'-m-1}$ yields a state E'' such that $h(N^{-1}(E'')) = (2^{2k} - m' - 1, 2^{2k} + m')$, and E'' is $\overline{\nu_2(2^{2k} - m')} = 1$-rooted embanked, which proves (1). This extra application of N' also increments a_1 by 2, so in total, a_1 has incremented by $2(m' - m - 1) + 2 = 2d$, while a_0 has decremented by $2(m' - m) = 2d$; this proves (2). $\square$

Starting with $S_k = (0, 2, 2^2, \ldots, 2^{2k}, 0)$, a brief computation yields $h(S_k) = (2^{2k}-1, 2^{2k})$ and $N(S_k) = S_k[2k+1]$. Thus, applying N' to S_k five times yields a 1-rooted embanked state $E := (N')^5(S_k)$ such that $h(N^{-1}(E)) = (2^{2k} - 3, 2^{2k} + 2)$. Hence, by inducting on m to reach $m = 2^{2k} - 2$, Proposition 4.1 and Proposition 3.4 imply the following corollary:

Corollary 4.2. There exists a rooted embanked state E such that $h(N^{-1}(E)) = (1, 2^{2k+1} - 2)$.

4.1. **Counting increments.** For rooted embanked states $E \mapsto E'$, let $\kappa_{E \to E'}(j)$ denote the number of $N\colon S \mapsto N(S)$ in $T'_{E \to E'}$ such that $N(S) = S[j]$.

Proposition 4.3. Suppose we are in the setting of Proposition 4.1, with E, E', m and m' defined as before. Then, for all $j \in [0, 2k+1]$, we have $\kappa_{E \to E'}(j) = \#\{e \in [m+1, m']\colon \max(1, 2^{j-1}) \mid e\}$.

Proof. For a statement X, define $\delta(X)$ to be 1 if X is true, and 0 if X is false. Fix a $j \in [0, 2k+1]$. By Corollary 3.8, we have

$$\begin{aligned}
\kappa_{E \to E'}(j) &= \begin{cases} \delta(\nu_2((m'-1)+1)+1 \geq j) + \#\{e \in [m+1, m'-1]\colon \nu_2(e) \geq j\} & j \equiv 0 \bmod 2 \\ \#\{e \in [m, m'-2]\colon \nu_2(e+1)+1 \geq j\} + \delta(\nu_2(m') \geq j) & j \equiv 1 \bmod 2 \end{cases} \\
&= \begin{cases} \delta(2^{j-1} \mid m') + \#\{e \in [m+1, m'-1]\colon 2^j \mid e\} & j \equiv 0 \bmod 2 \\ \#\{e \in [m+1, m'-1]\colon 2^{j-1} \mid e\} + \delta(2^j \mid m') & j \equiv 1 \bmod 2 \end{cases} \\
&= \begin{cases} \delta(2^{j-1} \mid m') + \#\{e \in [m+1, m'-1]\colon 2^{j-1} \mid e\} & j \equiv 0 \bmod 2 \\ \#\{e \in [m+1, m'-1]\colon 2^{j-1} \mid e\} + \delta(2^{j-1} \mid m') & j \equiv 1 \bmod 2 \end{cases} \\
&= \#\{e \in [m+1, m']\colon 2^{j-1} \mid e\},
\end{aligned}$$

where the third equality is using the fact that m and m' have odd 2-adic valuations, but every number in between has even 2-adic valuation. $\square$

4.2. **Endgame.** Our setup for the end of the proof is

$$S_k \to E \to E' \to E'',$$

where E, E' and E'' are all rooted embanked states satisfying $h(N^{-1}(E)) = (2^{2k} - 3, 2^{2k} + 2)$, $h(N^{-1}(E')) = (1, 2^{2k+1} - 2)$ and $h(N^{-1}(E'')) = (1, 2^{2k+1} - 1)$. In particular, we have $E' \to E''$ since Proposition 4.1(2) guarantees that E' will satisfy the bounds in Proposition 3.4 so that E' is weakly embanked, and then Corollary 3.8 implies that E'

is embanked. Also, we have $E = (N')^5(S_k)$. An explicit computation yields

$$\kappa_{S_k\to E}(j) = \begin{cases} 2 & j=0 \\ 3 & j=1 \\ 1 & j=2 \\ \overline{j} & 3\le j\le 2k+1, \end{cases}$$

$$\begin{aligned}\kappa_{E\to E'}(j) &= \#\{e\in[3,2^{2k}-2]\colon 2^{j-1}\mid e\} \\ &= \begin{cases} 2^{2k}-4 & j=0,1 \\ 2^{2k-1}-2 & j=2 \\ 2^{2k-j+1}-1 & j\in[3,2k] \\ 0 & j=2k+1, \end{cases}\end{aligned}$$

$$\kappa_{E'\to E''}(j) = \begin{cases} 1 & j=1 \\ 0 & j\neq 1. \end{cases}$$

It follows that

$$\begin{aligned}\kappa_{S_k\to E''}(j) &= \kappa_{S_k\to E}(j)+\kappa_{E\to E'}(j)+\kappa_{E'\to E''}(j) \\ &= \begin{cases} 2^{2k}-2 & j=0 \\ 2^{2k} & j=1 \\ 2^{2k-j+1}-1+\overline{j} & j\in[2,2k] \\ 1 & j=2k+1. \end{cases}\end{aligned}$$

Since moving from S to $S[j]$ increments a_j by 2, we obtain

$$\begin{aligned}a_j(E'') &= a_j(S_k)+2\kappa_{S_k\to E''}(j) \\ &= \begin{cases} 2^{2k+1}-4 & j=0 \\ 3\cdot 2^{2k} & j=1 \\ 3\cdot 2^{2k-j+1}-2+2\overline{j} & j\in[2,2k] \\ 2 & j=2k+1. \end{cases}\end{aligned}$$

Observe that E'' fails the conditions of Proposition 3.4, and so it will not make sense to apply N anymore. We resort to an explicit analysis from here. From the state E'', applying Zero yields

$$(0,0,3,3\cdot 2^1-2,3\cdot 2^2,3\cdot 2^3-2,\dots,3\cdot 2^{2k-1}-2,3\cdot 2^{2k},2^{2k+1}-5)(n,\sigma)=(2^{2k+1}-1,-1)$$

after which applying $2^{2k+1}-4$ Increment rules yields

$$(0,0,2^2,2^3-2,2^4,2^5-2,\dots,2^{2k-1}-2,2^{2k},2^{2k+1}-3,2^{2k+2}-2,-1) \qquad (n,\sigma)=(3,-1),$$

and now we Halve to obtain

$$(0,0,2^2,2^3-2,2^4,2^5-2,\dots,2^{2k-1}-2,2^{2k},2^{2k+1}-3,2^{2k+2}-2). \qquad (n,\sigma)=(1,+1).$$

Applying $2^{2k+2}-2$ Increment rules yields

$$(1,2,2^3,2^4-2,2^5,2^6-2,\dots,2^{2k}-2,2^{2k+1},2^{2k+2}-4,0) \qquad (n,\sigma)=(2^{2k+2}-1,+1)$$

after which we are forced to apply an Overflow rule to get

$$E_{k,\text{final}} := (0,2,2,2^3,2^4-2,\dots,2^{2k}-2,2^{2k+1},2^{2k+2}-4,0) \qquad (n,\sigma)=(0,-1).$$

By Proposition 3.4, $E_{k,\text{final}}$ is weakly embanked, and one may compute that $h(E_{k,\text{final}}) = (2^{2k+2}-1,2^{2k+2}-2)$. Further computing the state formed after the second Halve rule of $T_{E_{k,\text{final}}}$ shows that $E_{k,\text{final}}$ is in fact embanked with $N(E_{k,\text{final}}) = 2k+1$. By (1), applying $N'(\cdot)$ to $E_{k,\text{final}}$ yields

$$E'_{k,\text{final}} = (0,2,2^2,2^3,2^4,\dots,2^{2k},2^{2k+1},2^{2k+2}-2,0).$$

By Lemma 3.5, we have $h(E'_{k,\text{final}}) = (2^{2k+2} - 1, 2^{2k+2} - 1)$; therefore, Proposition 3.6 tells us that $N(E'_{k,\text{final}}) = E'_{k,\text{final}}[\nu_2(2^{2k+2} - 1) + 1] = E'_{k,\text{final}}[1]$. Hence,

$$N(E'_{k,\text{final}}) = (0, 2, 2^2, 2^3, 2^4, \ldots, 2^{2k}, 2^{2k+1}, 2^{2k+2}, 0),$$

which is just S_{k+1}.